Chapter 9. Charles

Networks

Definition.

$$N = (V, X, Y, A, C)$$

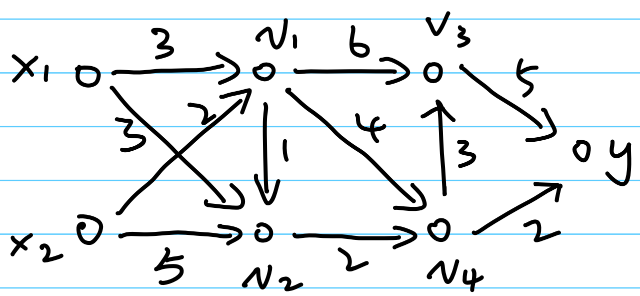
X: sources of N

Y: sinks of N

Other vertices are called intermediate vertices.

C: capacity function (≥ 0)

Example:



$X = \{x_1, x_2\}$

$Y = \{y\}$

$C = \{3, 2, \cdots\}$

Flow

Definition:

A flow in a network N is an integer-valued function defined on A such that:

<1> $0 \leq f(a) \leq c(a)$ for all $a \in A$

<2> $f^-(v) = f^+(v)$ for $v \in V - (X \cup Y)$

Zero flow: $\forall a \in A$, $f(a) = 0$

Definition:

The value of flow:

$$\mathrm{val} f = f^+(X) = f^-(Y)$$

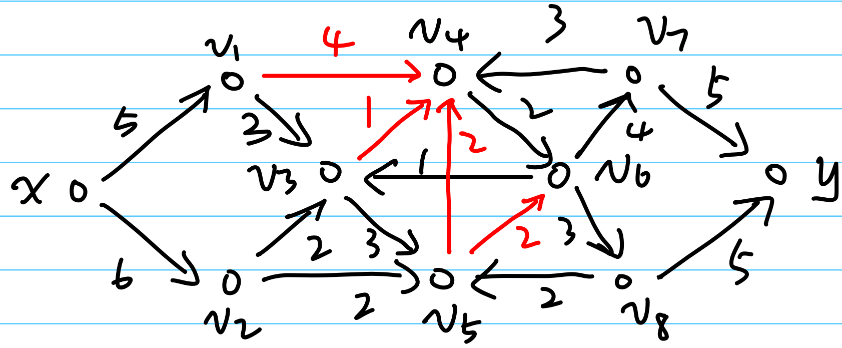
A flow f in N is a maximum flow if there is no flow f' in N such that $\mathrm{val} f' > \mathrm{val} f$.

Cuts

Definition:

Let N be a network with a single source x and a single sink y. A cut in N is a set of arcs of the form $(S, \bar{S})$, where $x \in S$, $y \in \bar{S}$.

Example:



$S = \{x, v_1, v_2, v_3, v_5\}$

$(S, \bar{S}) = \{v_1v_4, v_5v_4, v_5v_4, v_5v_6\}$

$\text{Cap}\,(S, \bar{S}) = 9$

Definition:

The capacity of a cut K is the sum of capacity of its arces.

$$\text{cap}\,K = \sum_{a \in K} c(a)$$

Lemma:

For any flow f and any cut $(S, \bar{S})$ in N

$$\text{val}\,f = f^+(S) - f^-(S)$$

Definition:

A cut K in N is a minimum cut if there is no cut K' in N such that $\text{cap } K' < \text{cap } K$.

Definition:

<1> $f(a) = 0$, f-zero

<2> $f(a) > 0$, f-positive

<3> $f(a) = c(a)$, f-saturated

<4> $f(a) < c(a)$, f-unsaturated

Theorem:

For any flow f and any cut $K = (S, \bar{S})$ in N

$$\text{val } f \leq \text{cap } K.$$

Equality holds if and only if each arc in $(S, \bar{S})$ is f-saturated and each arc in $(\bar{S}, S)$ is f-zero

Corollary:

f is a flow and K is a cut such that $\text{val} f = \text{cap } K$. then f is a maximum flow and K is a minimum cut.

Theorem (Ford, Fulkerson, 1956)

In any network, the value of a maximum flow is equal to the capacity of a minimum cut.

Maximum Flow Problem

Theorem:

A flow f in N is a maximum flow if and only if N contains no f-incrementing path.

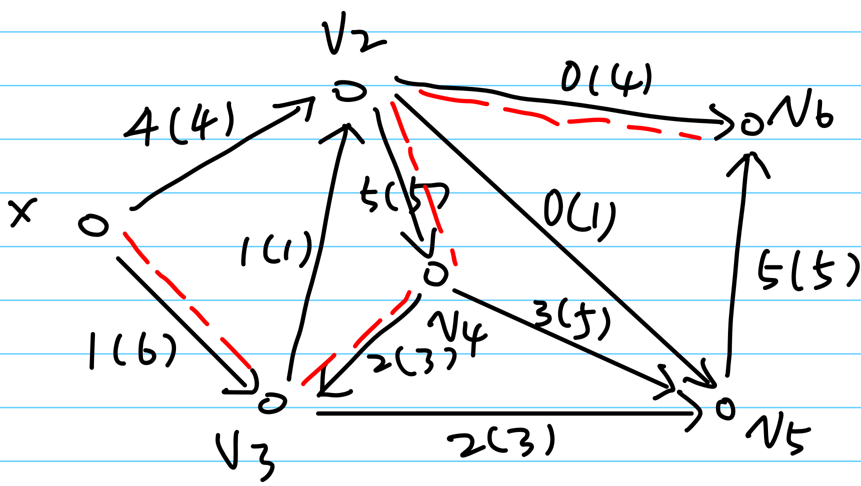
<1> Let $\Delta f(a) = \begin{cases} c(a) - f(a) & \text{forward arc} \\ f(a) & \text{reverse arc} \end{cases}$

$$\Delta f(P) = \min_{a \in P} \Delta f(a)$$

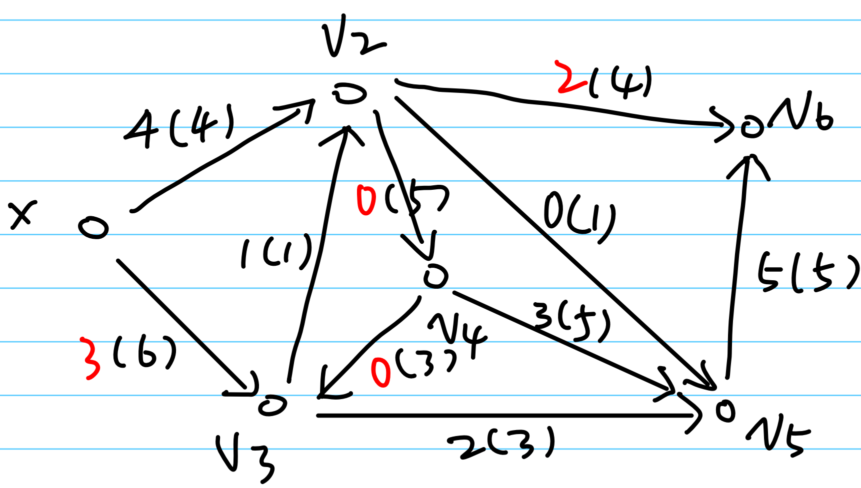
<2>

$$\hat{f}(a) = \begin{cases} f(a) + \Delta f(P) & \text{forward} \\ f(a) - \Delta f(P) & \text{Reverse} \\ f(a) & \text{Otherwise} \end{cases}$$

Example:



$\Delta f(P) = 2$



Labeling Procedure [illegible]

Minimum-Cost Flow [illegible]