Graphy Theory Chapter 6 2023/11/7

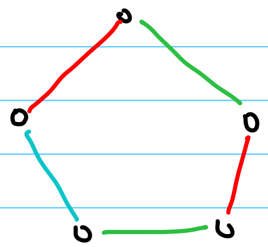
Edge Colouring

Proper edge k-colouring =

$c: E(G) \xrightarrow{Map} \{1,2,\cdots,k\}$, For every i, $c^{-1}(i)$ is Matching or ϕ.

$E_i = c^{-1}(i) = \{e \in E(G) \mid c(e) = i\}$. $(i=1,2,\cdots,k)$

Example



Definition

G $\exists$ one proper edge k-colouring, then G is edge k-colourable.

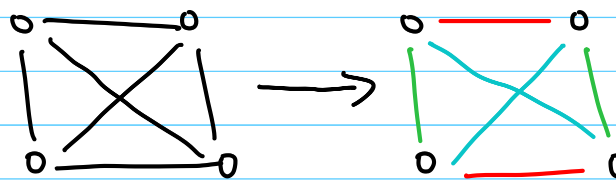
Edge Chromatic Number 边色数

$\chi'(G) = \min\{k \mid G \text{ is edge } k\text{-colourable}\}$.

<1> If $\chi'(G) = k$. $\{E_1, \cdots, E_k\}$ Every E_i is a Nonempty Matching.

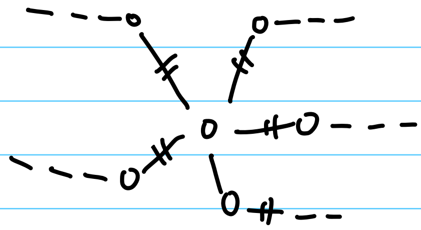
<2> $\chi'(K_{2n}) = 2n-1 = \Delta(K_{2n})$

Example



<3> $\chi'(G) \geq \Delta(G)$

Example

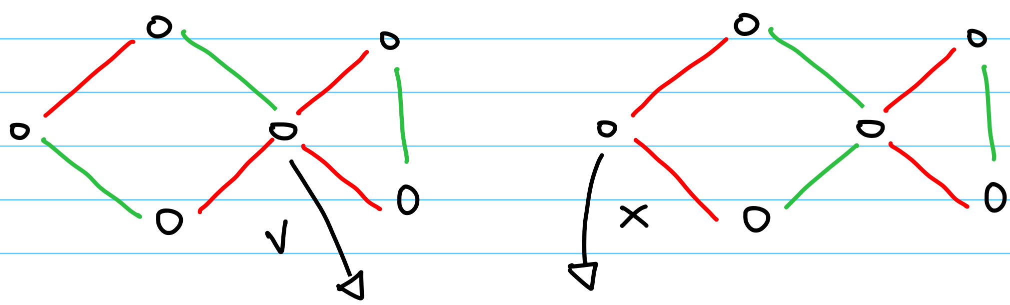


Lemma

G is a connected graph that is not an odd cycle. Then G has a 2-edge colouring in which both colours are represented at each vertex of degree ≥ 2.

Proof

<1> G is Euler Graph.



Start Point must $d(v) \geq 4$.

(2) Not an Euler Graph.

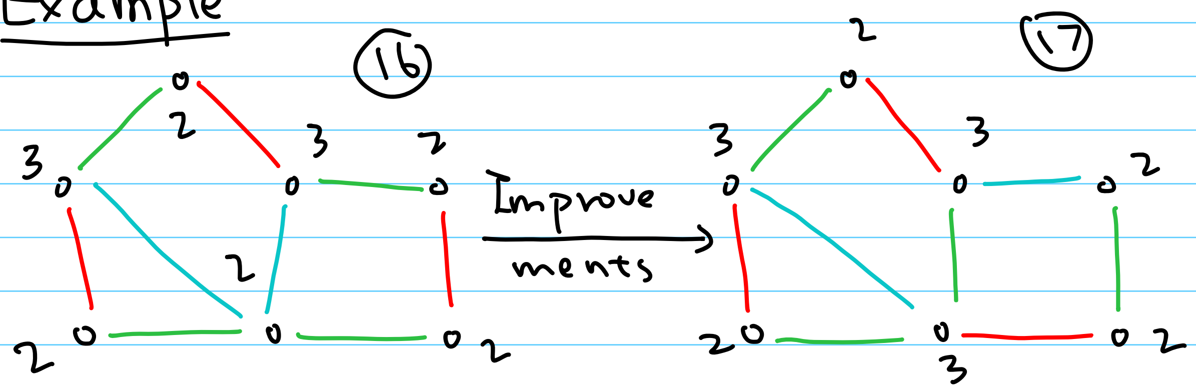
Adding a new vertex and joining it to each vertex of odd degree $\longrightarrow G'$

Optimal k-edge colouring

$c(v)$ denote the # of distinct colours at v

$$\sum_{v \in V(G)} c'(v) > \sum_{v \in V(G)} c(v)$$

Example



Lemma Let $\mathcal{C} = \{E_1, E_2, \cdots, E_k\}$ be an optimal k-edge colouring of G. If there is a vertex u in G and colours i and j such that i is not represented at u and j is respresented at least twice at u, then the component of $G[E_i \cup E_j]$ that contains u is an odd cycle.

Theorem (König, 1916)

G is bipartite, $\chi'(G) = \Delta(G)$

Theorem (Vizing, 1964)

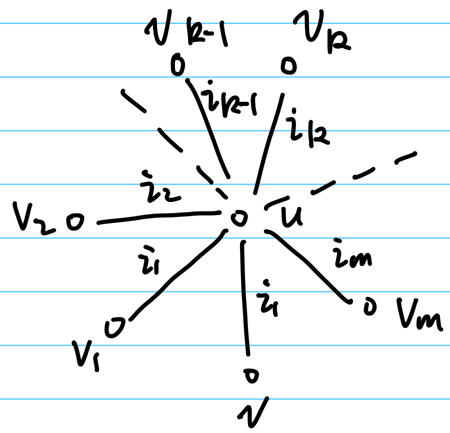
G is simple, then $\Delta(G) \le \chi'(G) \le \Delta(G)+1$

e.t. $\chi'(G) = \Delta(G)$ or $\Delta(G)+1$

Proof Only need to prove $\chi'(G) \le \Delta(G)+1$

If $\chi' > \Delta(G)+1$

$\exists\, i_1$ represented twice in u. and i_0 never represented in u. $(d(u) < \Delta+1)$



$\because d(v_1) < \Delta+1$, $\exists\, i_2$ not represtent in v_1

then i_2 must resprestent in u.

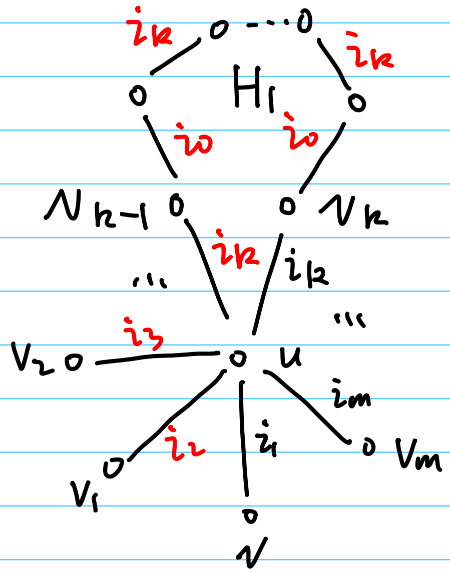
$\cdots$ we can find $\Rightarrow$

$\{v_1, v_2, \ldots, v_m\}$. $\{i_1, i_2, \cdots, i_m\}$

uv_j has i_j, and i_{j+1} not respresent in v_j. $\boxed{d(u) \text{ is finite.}}$

$\exists\, k < m$, $i_{m+1} = i_k$.

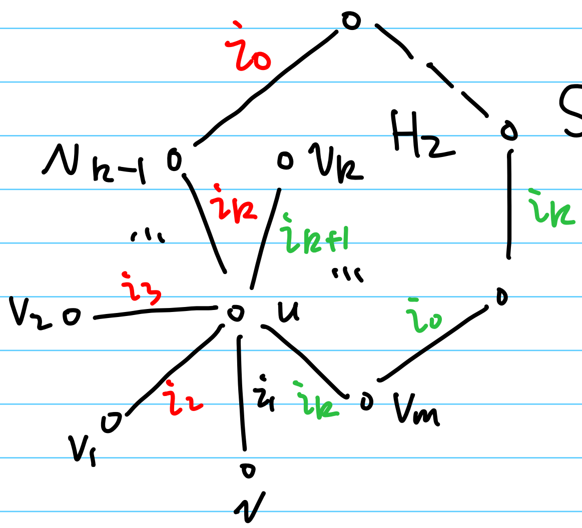
- Now for $1 \le j \le k-1$. Recolouring.



So H_1 is an odd cycle.

$d(v_k) = 2$

- Now for $k \le j \le m-1$. Recolouring



So H_2 is an odd cycle.

$d(v_k) = 1$

Contradiction: v_k has degree 1 in H_2

Graphy Theory Chapter.6 23/11/09

Vertex Colouring

Proper k-vertex colouring π is a map:

$$\pi : V(G) \longrightarrow \{1, 2, \cdots, k\}$$

$\pi^{-1}(i)$ is an independent set or ϕ

Chromatic Number

Definition:

$\chi(G) = \min\{k \mid G$ is vertex k-colourable$\}$.

Critical k-chromatic Graph

Definition

For a loopless Graph. $\chi(G) = k \ (k \geq 1)$.

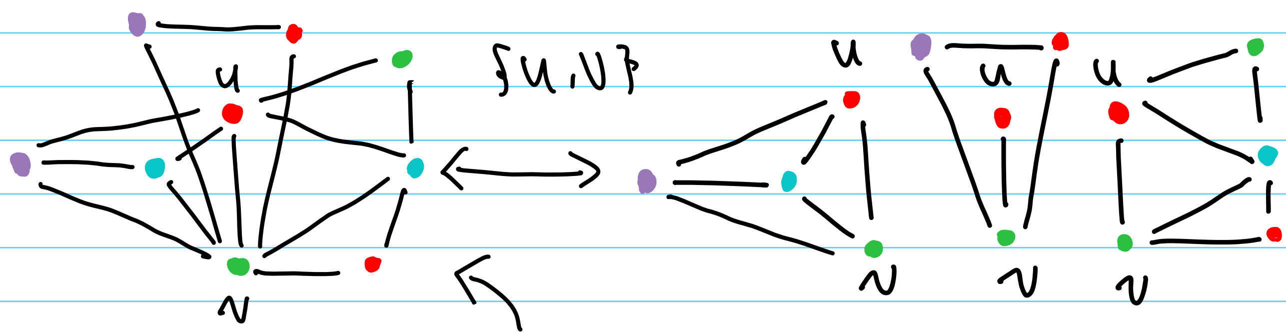
$\forall H \subseteq G$, $\chi(H) < k$. Then G is critical k-chromatic graph.

- G is 1-critical graph $\longleftrightarrow$ G is K_1.
- G is 2-critical graph $\longleftrightarrow$ G is K_2.
- G is 3-critical graph $\longleftrightarrow$ G is odd cycle.

Theorem

In a critical graph, no vertex cut is a clique.

Proof



Not a k-critical graph

Clique make sure vertex cut all have different colours.

Lemma

Every Critical graph is a block.

Theorem (Dirac, 1952)

G is k-critical ($k \geq 2$), Edge Connectivity $\kappa'(G) \geq k-1$.

Theorem (Brooks, 1941)

G is connected loopless simple graph and is neither an odd cycle nor a complete graph, then

$$\chi \leq \Delta(G)$$

Theorem

For any loopless Graph, All have

$$\chi(G) \geq \frac{\nu^2}{\nu^2 - 2\varepsilon}.$$

Theorem For any loopless G

(1) $\chi(G) + \alpha(G) \leq \nu + 1$

(2) $\chi(G) \cdot \alpha(G) \geq \nu$.

Theorem:

For Any loopless G.

$$\chi(G) \geq \omega(G) \leftarrow \text{Clique \#.}$$

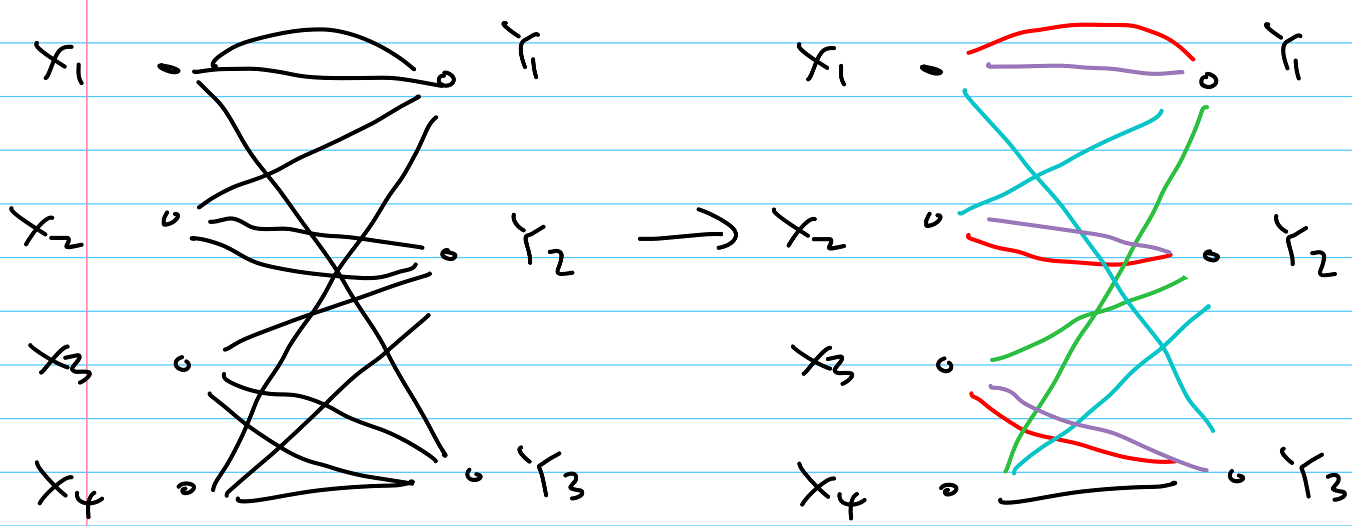
Edge Colouring Algorithm.

Timetableing Problem

m Teachers. $X_1, X_2, \cdots, X_m$

n classes. $Y_1, Y_2, \cdots, Y_n$

Teacher X_i is required to teach class Y_j for p_{ij} periods.



To partition Edges of G into as few Matchings as possible

Or

Properly colour the Edges with as few colours as possible

<u>Limitation</u>!

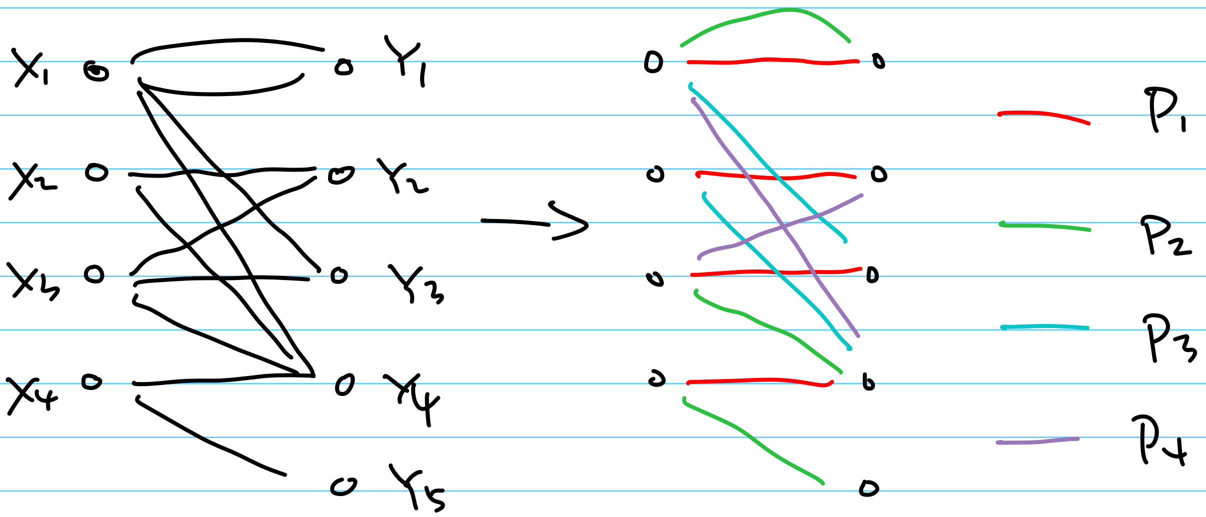
There are l lessons to be given, have been scheduled in a p-period timetable.

$\longrightarrow$ at least $\lceil l/p \rceil$ rooms are needed

<u>Example</u>:

$$T = \begin{array}{c c} & \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \text{ Classes} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Teacher



Teacher \ Period	1	2	3	4
x_1	y_1	y_1	y_3	y_4
x_2	y_2		y_4	
x_3	y_3	y_4		y_2
x_4	y_4	y_5		

$\downarrow$ Adjustment

Teacher \ Period	1	2	3	4
x_1	y_1	y_1	y_3	y_4
x_2	y_2		y_2	
x_3	y_3	y_4		y_2
x_4		y_5	y_4	

Vertex Colouring Algorithm

<1> 添边 米与合法

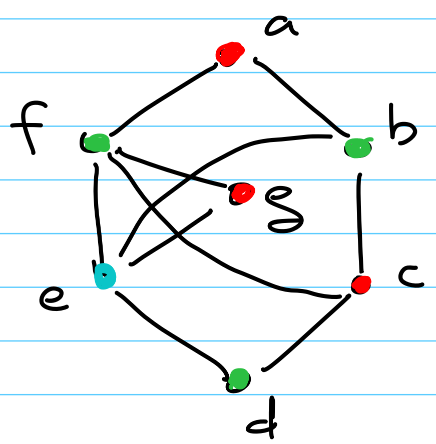
<2> canonical k-colouring.

Find 极大 Independent Sets

- Greedy Algorithm
- Not only one solution
- Approximate Algorithm

with $O(v^2)$

Example:



<1> $\{a, g, c\} = V_1$

<2> $G - V_1 \Rightarrow$

<3> $\{b, d, f\} = V_2$

<4> $G - V_1 - V_2$

<5> $V_3 = \{e\}$

$\Rightarrow$ Proper 3-vertex colouring

<3> Sequential Colouring.

Approximate Algorithm with $O(n^2)$

$V(G) = \{v_1, v_2, v_3, v_4 \cdots v_n\}$

$\boxed{\text{Results} \propto \text{Sequential}}$

Example:

$K_{n,n} = (X, Y) - \{x_i y_i\}_{i=1,\cdots,n} \longrightarrow G$

① $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n \to C = 2$

② $x_1, y_1, x_2, y_2, \cdots, x_n, y_n \longrightarrow C = n$

<4> Maximum Degree First

$V = \{v_1, \cdots, v_n \mid d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n)\}$

<5> Maximum Chromatic Degree First