Matching Theory

Definition

A subset M of E is called a matching in G if its elements are links and no two are adjacent in G; A matching M saturates a vertex v, and v is said to be M-saturated, if some edge of M is incident with v.

- If every vertex of G is M-saturated, the matching M is perfect.

Definition

An M-matching path in G is a path whose edges are alternately in $E \setminus M$ and M.

An M-augmenting path is an M-alternating path whose origin and terminus are M-unsaturated.

<u>Theorem</u> (Berge, 1957)
A matching M in G is a maximum matching <u>if and only if</u> G contains no M-augmenting path.

<u>Proof</u>:

$\rightarrow$ ✓

$\leftarrow$ Suppose M is not maximum

Let M' in G $|M'| > |M|$

Let $H = G[M \oplus M']$ which is a edge-induced subgraph.

Every vertex of H has degree either 1 or 2 in H. Thus each component of H is either an even cycle with edges

alternately in M and M', or else a path with edges alternately in M and M'.
$\because |M'| > |M|$, therefore some path component P of H must start and end with edges of M'.
Thus $\exists$ an M-augmenting path. ▨

Perfect Matching

Definition

A component of a graph is odd or even according as it has an odd or even number of vertices. We denote by $o(G)$ the number of odd components.

Theorem (Tutte, 1947)

G has a perfect matching if

and only if $o(G-S) \le |S|$ for all $S \subset V$.

<u>Corollary</u> (Peterson, 1891)

Every 3-regular without cut edges has a perfect matching.

<u>Matching in Bipartite</u>

<u>Theorem</u> (Hall, 1935)

Let G be a bipartite graph with bipartite (X, Y). Then G contains a matching that saturates every vertex in X if and only if
$$|N(S)| \ge |S| \text{ for all } S \subseteq X$$

<u>Definition</u>

A k-factor of a graph is a spanning k-regular subgraph.

a k-factorization partitions the edges of the graph into disjoint k-factors.

- A graph G is said to be k-factorable if it admits a k-factorization.

Algorithm

To find maximum matching in Bipartite graph.

- Hungarian Algorithm

To find maximum weights matching in weighted Bipartite Graph.

- Kuhn-Munkers Algorithm