Chapter 2. Connectivity by Charles

Cut Vertex and Cut Edge

Definition

$v \in V(G)$. If $\omega(G-v) > \omega(G)$, then v is a cut vertex of G.

(ω is the # of connected components)

- 环边的顶点不算.

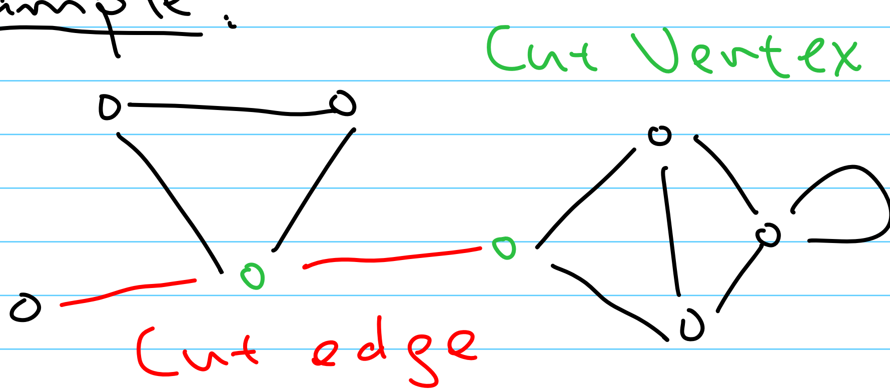
Theorem

v is cut vertex of tree T if and only if $d(v) > 1$

Definition

$e \in E(G)$. If $\omega(G-e) > \omega(G)$, then e is a cut edge of G.

Example:



Theorem.

Edge e is cut edge of G if and only if e is contained in no cycle of G

Theorem:

A connected graph is tree if and only if its every edge is cut edge.

Connectivity

Definition:

A vertex cut of G is a subset V' of V such that $G - V'$ is disconnected. A k-vertex cut is a vertex cut of k elements. The connectivity $\kappa(G)$ of G is the minimum k for which G has k-vertex cut.

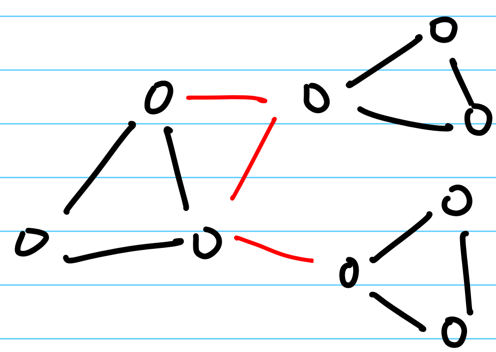
- Cut vertex is 1-vertex cut
- Complete graph has not cut vertex

If $\kappa(G) \geq k$, then G is called k-connected

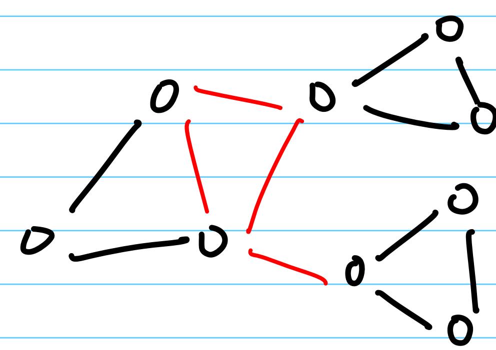
<u>Definition.</u>

An edge cut of G is a subset of E of the form $[S, \bar{S}]$, where S is a nonempty proper subset of V. A k-edge cut a cut edge of k elements. The edge-connectivity of G is the minimum k for which G has k-edge cut: $\kappa'(G)$

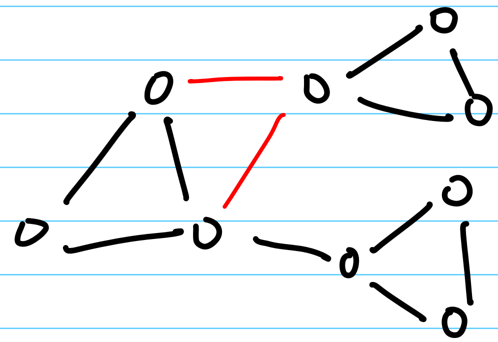
<u>Example</u>



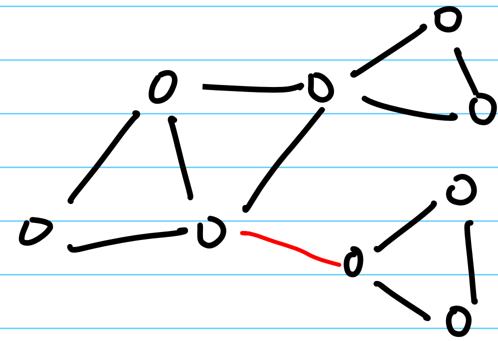
Edge Cut



Not edge Cut



Minimal E-C



Minimum E-C

Theorem

$$\kappa(G) \le \kappa'(G) \le \delta(G)$$

Theorem

Connected Graph with ν vertices and ε edges, we have $\kappa(G) \le \left\lfloor \frac{2\varepsilon}{\nu} \right\rfloor$

Corollary

G is a simple Graph, if $\delta(G) \ge \frac{\nu-1}{2}$, then G is connected graph.

Block

Definition:

A connected graph that has no cut vertices is called a block. A block of a graph is a subgraph that is a block and is maximal with respect to this property.

- Every block with at least three vertices is 2-connected.

<u>Theorem</u> (Whitney, 1932)

If G is 2-connected, then any two vertices of G <u>lie on a common cycle</u>. (are connected by at least two internally-disjoint paths.)

<u>Proof</u>.

$\Leftarrow$ Easy

$\Rightarrow$ 归纳法

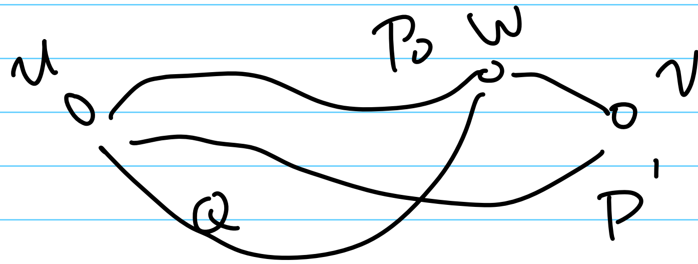
$d(u,v) = 1$. $G - uv$ is connected because $\kappa' \geq \kappa \geq 2$. $\exists$ Path P in $G - uv$. $\therefore \exists$ cycle $P + uv$

Assume $d(u,v) < k$ ✓

Let's prove $d(u,v) = k$. ✓

$P_0 = u \cdots w v$ with k length

$\therefore d(u,w) = k-1$ $\therefore u, w$ in a cycle.



$\because G - \{w\}$ connected.

$\therefore \exists P'(u,v)$

We can find such a cycle ✓ ∎

<u>Corollary.</u>

If G is a block with $\nu \geq 3$, then any two edges of G lie on a common cycle.